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ON TWO-DIMENSIONAL SUPERSONIC FLOWS

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SUMMARY

In previous reports by the author, the method of operators has been applied to the investigation of two-dimensional irrotational flow patterns of an ideal compressible fluid, mostly in the subsonic case. Using a similar approach in the present paper, the author defines two different operators which generate supersonic two-dimensional flow patterns from differentiable functions of one real variable. (An operator is any rule by means of which one function is converted into another. This concept is discussed in some detail in the body of this paper.) Since operators of the type considered preserve many properties of the functions to which they are applied and since the theory of functions of one real variable is more extensively developed than that of solutions of the compressibility equations, the results obtained can be used as a basis for the investigation of supersonic flow patterns.

INTRODUCTION

By following the line of approach developed in previous publications for the study of two-dimensional subsonic flows (see references 1 to 4), the method of operators is applied in the present paper for generating two-dimensional steady supersonic flow patterns. Using the hodograph method introduced by Chaplygin, the author considers the stream function in the Λ,θ-plane, where

$$\Lambda = h^{-1}\text{arc tan}\left[h\left(M^2 - 1\right)^{1/2}\right] - \text{arc tan}\left[\left(M^2 - 1\right)^{1/2}\right]$$

(see equations (5) and (6)), M being the local Mach number; θ, the angle between the velocity vector and the x-axis; and h, a constant characteristic of the fluid.[1]

The stream functions, considered as a function of Λ and θ, satisfy a linear differential equation

$$\psi_{\Lambda\Lambda} - \psi_{\theta\theta} + 4N_1\psi_\Lambda = 0$$

Two operators which generate solutions of this equation are defined in the present paper. The first operator is obtained by using Riemann's function. (See section entitled "Operator Obtained by the Use of Riemann's Function.") The second operator is derived by means of the theory of integral operators. In the section dealing with the second operator, a set of functions $H(\Lambda)$, $E_m{}^n(\Lambda)$, $n = 1, 2, \ldots$, are determined such that

$$P(f) = \lim_{m \to \infty} \left\{ H(\Lambda) \left[f(\xi) + \sum_{n=1}^{\infty} E_m{}^{(n)}(\Lambda) f^{[n]}(\xi) \right] \right\}$$

[1]At every point (x,y) of a steady two-dimensional flow the velocity vector, the Cartesian components of which are $q \cos \theta$ and $q \sin \theta$, is defined. If $\psi(x,y)$ denotes the stream function and $\rho(x,y)$, the density, the coordinates x and y can be expressed as functions of q and θ (and hence, using relations (5) and (6) of Λ and θ) by solving the equations

$$q \cos \theta = \rho^{-1} \frac{\partial\psi}{\partial y}$$

and

$$q \sin \theta = -\rho^{-1} \frac{\partial\psi}{\partial x}$$

Substituting these expressions for x and y into $\psi(x,y)$, the function $\tilde{\psi}(\Lambda,\theta) = \psi[x(\Lambda,\theta), y(\Lambda,\theta)]$ is obtained, i.e., the stream function in the Λ,θ-plane.

where

$$f^{[n]}(\xi) = \int_a^{\xi} f^{[n-1]}(\xi)\, d\xi$$

$$f^{[0]}(\xi) = f(\xi)$$

$$\xi = \Lambda + \theta \qquad \text{or} \qquad \xi = \Lambda - \theta$$

(where f is an arbitrary differentiable function of one real variable ξ) represents a solution of the equation

$$\psi_{\Lambda\Lambda} - \psi_{\theta\theta} + 4N_1\psi_{\Lambda} = 0$$

so that $P(f)$ can be interpreted as the stream function of a supersonic flow. The equation of a streamline of the flow in the physical plane is then given in parametric form:

$$x = \int\left(\frac{1}{\rho}\,\psi_{\Lambda}\cos\theta\,\frac{d\Lambda}{dq} - \frac{1}{\rho q}\,\psi_{\theta}\sin\theta\right)d\theta$$

$$+ \left[\frac{\left(M^2 - 1\right)}{\rho q^2}\,\psi_{\theta}\cos\theta\,\frac{dq}{d\Lambda} - \frac{1}{\rho q}\,\psi_{\Lambda}\sin\theta\right]d\Lambda$$

$$y = \int\left(\frac{1}{\rho}\,\psi_{\Lambda}\sin\theta\,\frac{d\Lambda}{dq} + \frac{1}{\rho q}\,\psi_{\theta}\cos\theta\right)d\theta$$

$$+ \left[\frac{\left(M^2 - 1\right)}{\rho q^2}\,\psi_{\theta}\sin\theta\,\frac{dq}{d\Lambda} + \frac{1}{\rho q}\,\psi_{\Lambda}\cos\theta\right]d\Lambda$$

where $\psi(\Lambda,\theta) = c = \text{Constant}$. The representation $P(f)$ is investigated in the present paper, and, in particular, methods for the actual

determination of $E_m^{(n)}$ are discussed as well as the domain in the Λ,θ-plane where the order of summation and passage to the limit can be inverted.

The evaluation of the operators, that is, the determination of the flow pattern when the function f is given, involves lengthy computations. However, in a manner similar to that described in reference 4 for the subsonic case, it is possible to prepare, once and for all, tables which greatly facilitate numerical computation of flow patterns.

The results obtained in the present paper can be used as a basis for the investigation of supersonic flows.

The next problem which arises in this connection consists in determining for a desired flow the functions of one variable which have to be inserted into the operator in order to obtain the desired boundary shape or to decide that such a flow does not exist. The author expects to treat these problems in subsequent publications. See also the SUMMARY REMARKS presented at the end of the present paper.

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SYMBOLS

a	velocity of sound in compressible fluid
a_o	velocity of sound at a stagnation point
$h = \sqrt{\frac{k-1}{k+1}}$	
k	ratio of specific heat at constant pressure to specific heat at constant volume ($k = 1.4$ for air)
q	speed of a fluid particle $\left(\sqrt{u^2 + v^2}\right)$
u	x-component of velocity vector ($q \cos \theta$)
v	y-component of velocity vector ($q \sin \theta$)
x,y	Cartesian coordinates in physical plane

$B = \sqrt{M^2 - 1}$ for $M > 1$

M Mach number (q/a)

$R(\xi,\eta;\xi_o,\eta_o)$ Riemann's function for equation (15)

$T = \sqrt{1 - M^2}$ for $M < 1$

$\tilde{\beta} = T + iB$

$l = \lambda + i\Lambda$

$E^{(n)}(\tau)$ functions defined by recurrence relations (22)

$F_1(2\Lambda)$ see equation (16)

$F_2(2\Lambda')$ see Theorem under section Integral Operator of Type Given by Formula (11)

$F_2^{(m)}(2\Lambda')$ a sequence of functions converging to $F_2(2\Lambda')$; see equation (40)

α_n coefficients of Taylor series expansion of $F_2(2\Lambda')$; see equation (39)

$\xi = \Lambda + \theta$

$\eta = \Lambda - \theta$

$\xi' = \Lambda' + \theta$

$\eta' = \Lambda' - \theta$

$$\lambda = \frac{1}{2}\left(\log\frac{1 - T}{1 + T} + \frac{1}{h}\log\frac{h^{-1} + T}{h^{-1} - T}\right) \text{ for } M < 1;\ \log = \log_e$$

$$\Lambda = \frac{1}{h}\arctan(hB) - \arctan B \text{ for } M > 1$$

$$\Lambda' = \Lambda - \frac{1}{2}\pi\left(\frac{1}{h} - 1\right)$$

θ angle between velocity vector and positive direction of x-axis (arc tan v/u)

ϕ potential function

ψ	stream function
ψ^*	"reduced stream function" (see equation (14))
$\psi^{*(m)}$	a sequence of functions converging to ψ^*; see equation (37)
ρ_o	density of fluid at a stagnation point
$\tilde{\rho}$	density of fluid in motion
ρ	"reduced density" $\left(\tilde{\rho}/\rho_o\right)$
N_1	see equation (9)

Note.– Frequently quantities are considered as functions of different pairs of variables, so that different symbols should be used to designate the functional dependence in the various planes. However, this is not done herein, the same symbol being used in each plane. This should cause no confusion, for the meaning is clear from the context.

ANALYSIS

The Mathematical Problem in Simplified Form in the Case of a Supersonic Motion of a Compressible Fluid

In this report, operator methods are applied to the problem of constructing two-dimensional, irrotational, steady flow patterns of a compressible fluid at supersonic velocities.

Just as in the case of an incompressible fluid, the steady, irrotational flow is completely described by either the velocity potential $\phi(x,y)$ or the stream function $\psi(x,y)$. If it is assumed that the motion is adiabatic and that the fluid is a polytropic gas, then the motion is governed by the following system of nonlinear partial differential equations:

$$\left.\begin{aligned} \phi_x &= \rho^{-1}\psi_y \\ \phi_y &= -\rho^{-1}\psi_x \\ \phi_x &= \frac{\partial \phi}{\partial x} \\ &\cdots\cdots \end{aligned}\right\} \qquad (1)$$

and

$$\rho = \rho(x,y) = \left[1 - \frac{k-1}{2}\left(\frac{\phi_x{}^2 + \phi_y{}^2}{a_o{}^2}\right)\right]^{\frac{1}{k-1}}$$

Here $\rho = \tilde{\rho}/\rho_o$ is the "reduced density" of the fluid, that is, the density $\tilde{\rho}$ divided by the density ρ_o at rest; x,y are Cartesian coordinates in the physical plane, that is, in the plane where the motion takes place; k is the ratio of the specific heat at constant pressure to the specific heat at constant volume; and a_o is the velocity of sound in the fluid at rest. (See reference 1, equations (24) and (25).)

A fluid motion satisfying equation (1) may show two essentially distinct types of behavior, according as the speed of the fluid is greater or less than the speed of sound in the fluid. The mathematical reason is readily seen in the following manner. If ρ and ψ are eliminated from equation (1) the result is a single, nonlinear, partial differential equation of the second order for ϕ:

$$\left(a^2 - \phi_x{}^2\right)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + \left(a^2 - \phi_y{}^2\right)\phi_{yy} = 0 \tag{2}$$

where

$$a^2 = a_o{}^2 - \tfrac{1}{2}(k-1)\left(\phi_x{}^2 + \phi_y{}^2\right) \tag{3}$$

is the square of the local velocity of sound. (See equation (28) of reference 1.) The discriminant of equation (2) is

$$\begin{aligned} &\phi_x{}^2\phi_y{}^2 - \left(a^2 - \phi_x{}^2\right)\left(a^2 - \phi_y{}^2\right) \\ &\quad = -a^2\left(a^2 - \phi_x{}^2 - \phi_y{}^2\right) = a^2\left(q^2 - a^2\right) \end{aligned} \tag{4}$$

where q^2 is the speed at the point x,y under consideration. In regions where the velocity is subsonic, that is, $q < a$, this equation is of elliptic type, whereas in regions where the velocity is supersonic, that is, $q > a$, equation (2) is of hyperbolic type. (See reference 5, pp. 1–4.)

The solutions of these two types of differential equation have markedly different functional character, and these differences are, of course, reflected as the two different types of compressible fluid motion.

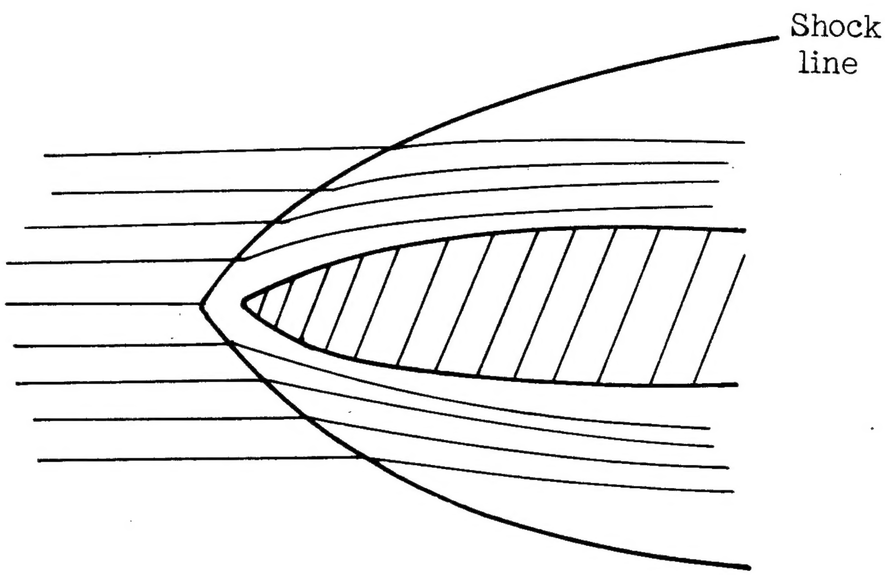
In spite of these differences, however, the operators which have been used for the solution of subsonic flow problems may be adapted to the solution of supersonic problems after certain formal modifications. Before this is done, however, it is necessary to make a few preliminary remarks.

The differential equations (1), both for subsonic and supersonic velocities, were derived under rather restrictive assumptions, namely, that the flow is two-dimensional, that the fluid is inviscid, and that the motion is irrotational and steady. By these assumptions the problem is considerably idealized; but experience has shown that in a great many important cases occurring in practice, when the speed is small, the consequences of the theory are in satisfactory agreement with observations. Moreover, a physical justification for the apparently drastic assumption that the motion is laminar is contained in the Von Mises "hydraulic hypothesis." (See reference 6, p. 84.)

In many instances where the indicated idealization is not admissible, the "idealized" theory can at least be used as a first approximation, which can be improved by correction terms. So, for instance, the boundary-layer theory describes the motion near the boundary where viscosity effects cannot be neglected, whereas in the interior of the flow pattern the theory of ideal, inviscid fluid can be successfully used without any alteration in the case of a comparatively slow motion.

Unfortunately, when the speed increases considerably, the situation changes completely, and the theory of an incompressible fluid can no longer be used as a first approximation. Furthermore, at supersonic velocities a completely new phenomenon appears, the influence of which is not easily assessed. When a compressible fluid flows around an immersed body at supersonic velocities, a very characteristic phenomenon is the appearance of a shock line. It appears as a curve (or curves) which divides the physical plane into two (or more) unconnected infinite regions. In the neighborhood of the shock line, the fluid may no longer in general be regarded as nonviscous; there is a discontinuity of velocity along the shock line; and in the region of the physical plane beyond the shock line (except in a few special cases) the flow is no longer irrotational. For all these reasons, the differential equations (1) may no longer be supposed to represent the actual physical situation. Exact equations can be derived, of course, but they are extremely complicated nonlinear partial differential equations for which no mathematical theory now exists; and the problem presented (which is

neither an initial-value problem nor a boundary-value problem, because the shock line is not prescribed) cannot be handled by any known mathematical means.



An example of a supersonic flow pattern around an obstacle with a shock line.

It can be expected that in the same manner as the idealized theory of an incompressible fluid represents an excellent "first approximation" in the sense just explained for comparatively slow motions, the idealized theory of compressible fluid will represent a similar "first approximation" for motions with considerable speed[2] (if necessary, with certain additional alterations).

In this and previous papers on the subject by the author, therefore, no attempt is made to find exact solutions of the actual physical problem. Instead it is assumed that the part of the flow around the obstacle satisfies the conditions under which equations (1) were derived. The method of operators is then used to find and investigate exact solutions of this idealized problem. The results may be regarded as first approximations to the true facts in the sense just explained.

[2]It is clear that for many purposes it will be necessary to develop a three-dimensional theory. This theory, even in the case of an incompressible fluid, is only in the very first stages of development. The theory of operators and some other modern mathematical means, as the author expects to show in another paper, will make it possible to deal with certain problems in the three-dimensional case. These methods, besides yielding results for subsonic motions of a compressible fluid, will also yield results of certain problems (as yet unsolved) for an incompressible fluid.

The question may also be regarded from another viewpoint. It would often be possible to find a numerical solution of a specific problem, but such a result has little theoretical value. A satisfactory theory must be capable of doing much more. It must be able to give at least qualitative information concerning how the motion of the fluid is affected by variations in the form of the immersed body, the specific properties of the fluid, or any other significant variables; and preferably it should also be capable of providing quantitative predictions. The operator method of solving differential equations has the advantage that it may be used to investigate the functional properties of the solutions it yields. (For details, see references 2 and 3.)

Remark.– From the purely mathematical point of view, an operator is a rule which is applied to a function of an appropriate class (usually denoted as class A) to obtain a function belonging to another class (class C). A simple example of an operator is the process of obtaining a harmonic function of two real variables (class C) by taking the real (or imaginary) part of an analytic function of a complex variable (class A). This operator preserves many properties of analytic functions and is often used as a tool for investigation of harmonic functions.

A similar relationship may be seen in the operators employed in the theory of compressible fluid. These operators appear in the form (equation (11)) for subsonic flows and in a similar form for supersonic flows. In the case of subsonic flows, equation (11) acts on analytic functions of a complex variable (class A) and produces complex solutions of equation (10) (class C). In the supersonic case, the operator acts on a differentiable function of a real variable (class A') and produces solutions of equation (9) (class C'). These operators are special cases of operators introduced by the author which convert analytic functions of a complex variable into complex solutions of a given linear differential equation of elliptic type and differentiable functions of a real variable into solutions of a given equation of hyperbolic type.

The theory of integral operators gives a deep insight into the property of the solutions of the compressibility equation. As indicated in references 1, 2, and 3, solutions of equation (10) can be written in the form

$$\psi = \int_{-1}^{+1} E(M,\theta,t)f\left\{\frac{1}{2}\left[\lambda(M) + i\theta\right]\left(1 - t^2\right)\right\} dt/\sqrt{1 - t^2}$$

(See equation (55) of reference 1.) Here $\lambda(M)$ is a function of the Mach number M, which in the subsonic case is real, and in the supersonic case purely imaginary. The expression $E(M,\theta,t) = \tilde{E}(\lambda,\theta,t)$ is a function which, in the "simplified case" (Tricomi case), when considered as a function of $u = t^2(\lambda + i\theta)/2\lambda$, satisfies the hypergeometric equation. In the case of the "exact" compressibility equation, behavior and properties of $\tilde{E}(\lambda,\theta,t)$ are completely analogous to those in the simplified case. It is described in sections 4 and 5 of reference 3.

The operator method[3] has another advantage which is very important from the practical point of view. All too frequently the only thing which stands in the way of solving an engineering problem mathematically is the difficulty of performing the incidental numerical computations. Although the use of modern automatic computing machinery in many instances reduces considerably the amount of labor needed, it still remains a serious obstacle. Whenever possible, the ultimate numerical evaluation should be kept in mind when the mathematical analysis is carried out. It is a definite advantage when the computations required by a theory may be readily performed. It will be seen that when hydrodynamical problems are attacked by the operator method, the formulas which occur contain a number of standard functions which are the same in all problems of the same type. Consequently these functions may be computed and tabulated once and for all. When this has been done, the computations required for any specific problem are much shorter and simpler.

The Method of Operators in the Theory of Compressible Fluids

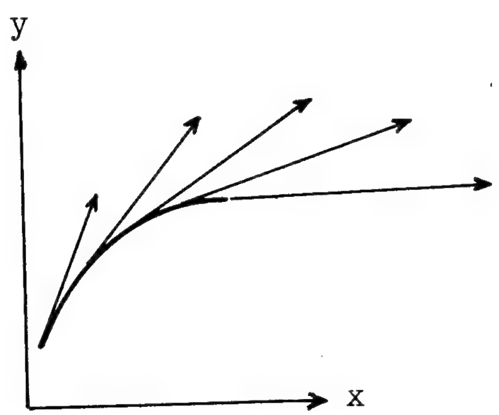
The idea of the method of operators is now explained briefly.

It has already been stated that the partial differential equations of compressibility are nonlinear and that a mathematical theory of such equations does not yet exist. Fortunately, it is possible to linearize the equations of motion of a compressible fluid by a change of variables. Therefore, hydrodynamical problems are not studied in the physical plane where the flow occurs but must be investigated in some auxiliary plane where the differential equations are easier to handle. This entails considerable distortion of the flow pattern and a consequent loss of intuitive appeal.

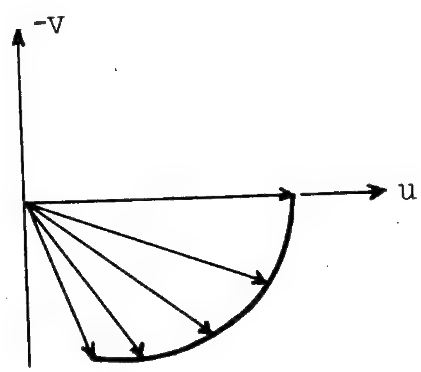
The process of linearization is conveniently performed in two stages. In the first place it is noted that, in general, in a sufficiently small portion of the flow region, a particle may be just as unambiguously specified by its velocity vector as by its position, although this will

[3]The main mathematical idea of the operator method consists in employing certain integral operators which transform functions of one complex (or real) variable into "classes" of complex solutions of given linear partial differential equations and preserves many properties of the class of functions to which the operator is applied. Thus, in the case of equations of elliptic type, the application of the operators results in the derivation of complex solutions of the given solution's equation which in many respects behave like analytic functions of a complex variable and help in the investigation of real solutions in a manner which bears a close analogy to the role of analytic functions of a complex variable for real harmonic functions in two variables.

not generally be true over the flow as a whole[4]. The first change of variables is, therefore, from the physical plane with coordinates x,y to the so-called velocity or hodograph plane in which the polar coordinates q,θ are used, q being the speed of the particle, and θ, the angle between the direction of motion of the particle and the positive x-axis. This transformation obviously introduces distortion (see following figs.) and it is by no means bi-unique,



Motion in the physical plane.



The image in the hodograph plane of the flow indicated in figure at left.

since it usually happens that the velocity vectors are equal at different points of the flow region. A further distortion is now produced by plotting $\log q$ and θ as Cartesian coordinates (logarithmic plane). Finally, the substitution

$$\Lambda = \Lambda(q) = \frac{1}{h} \text{ arc tan } \left[h\left(M^2 - 1\right)^{1/2}\right] - \text{arc tan } \left[\left(M^2 - 1\right)^{1/2}\right] \tag{5}$$

(see fig. 1) where

$$\left.\begin{aligned} M &= \frac{q}{a} \\ h &= \left(\frac{k - 1}{k + 1}\right)^{1/2} \end{aligned}\right\} \tag{6}$$

[4]In order to make the correspondence between the flow pattern in the physical plane and its image in the hodograph plane 1-to-1, it is necessary in this case to locate the hodograph on a Riemann surface with the proper number of sheets.

has the effect of changing the scale along the log q-axis (pseudo-logarithmic plane). This introduces an additional distortion but succeeds in reducing the differential equation for the potential function to the form

$$\phi_{\Lambda\Lambda} - \phi_{\theta\theta} - 4N_1\phi_{\Lambda} = 0 \tag{7}$$

where

$$N_1 = \frac{k+1}{8} \frac{M^4}{\left(M^2 - 1\right)^{3/2}} \tag{8}$$

The stream function satisfies the differential equation

$$\psi_{\Lambda\Lambda} - \psi_{\theta\theta} + 4N_1\psi_{\Lambda} = 0 \tag{9}$$

In the subsonic case, a similar procedure leads to the differential equation

$$\psi_{\lambda\lambda} + \psi_{\theta\theta} + 4N\psi_{\lambda} = 0 \tag{10}$$

(see equation (47) of reference 1), where $\lambda = \lambda(q)$ is a function of the speed alone, the form of which is very similar to that of $\Lambda(q)$ in equation (5). The operator method of finding subsonic compressible flow patterns is now easily described. It is based on making suitable use of the following formula for generating solutions of equation (10):

$$\left.\begin{aligned} \psi(\lambda,\theta) &= \operatorname{Im} P(g) \\ P(g) &= H(2\lambda)\Bigg[g(\zeta) \\ &\quad + \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\, n!} Q^{(n)}(2\lambda) \int_0^{\zeta} \cdots \int_0^{\zeta_{n-1}} g(\zeta_n)\,d\zeta_n \cdots d\zeta_1\Bigg] \end{aligned}\right\} \tag{11}$$

where $\zeta = \lambda - i\theta$. (See reference 1, formula[5] (66).) In this formula g represents an arbitrary analytic function of a complex variable ζ

[5]Note that in formula (66) of reference 1, there is a misprint: the factor $\frac{(2n)!}{2^n\, n!}$ should be replaced by $\frac{(2n)!}{2^{2n}\, n!}$.

which is regular at $\zeta = 0$; whereas H and the $Q^{(n)}$'s are certain functions of λ only and are the same for all stream functions. They may be computed once and for all and the results used in all future hydrodynamical problems. (See reference 4.)

The foregoing formula is employed in the following manner. If any analytic function g of a complex variable (regular at $\zeta = 0$) is substituted into formula (11), the result is a stream function of a subsonic flow pattern. On the other hand, it is well known that an analytic function $g(\log q - i\theta)$ may be regarded as defining a flow pattern of an incompressible fluid. Consequently formula (11) may be interpreted as an operator which distorts an incompressible-flow pattern into a compressible one. (It may be said that the operator acts as a distorting mirror which reflects an incompressible-flow pattern as a subsonic compressible-flow pattern.) The operator formula is of such a form that a knowledge of the incompressible-flow pattern used leads directly to both qualitative and quantitative knowledge of the behavior of the stream function of the corresponding compressible flow.

An obvious objection is that, whereas any incompressible flow will lead to some compressible flow when a specific obstacle is prescribed, there seems to be no indication of how to find precisely the g to be substituted into the operator. A procedure to solve this problem approximately can be developed (see reference 7), but this is not considered herein. However, it is easy to see why the same analytic function $g(\zeta)$ which solves the corresponding incompressible problem can be expected to yield a first approximation to the compressible problem in the case of low speeds (i.e., small Mach number); for in such a case the solution (formula (11)) is easily shown to yield a flow, very similar to the incompressible flow, which agrees with the physical state of affairs.

The operator method as just described involves the following elements:

(1) It is necessary to have considerable knowledge about incompressible-flow patterns, in particular about their images in the hodograph and logarithmic planes.

(2) The operator formula has to be derived, the functions H and $Q^{(n)}$ have to be determined, and rules must be developed for interpreting properties of incompressible-fluid motions as properties of the compressible flows given by the operator.

(3) After a solution of a problem on compressibility has been found in terms of the variables (λ,θ) (i.e., in the so-called pseudo-logarithmic plane), means must be furnished for determining the actual flow in the physical plane.

It is obvious that one reason why this method can be so powerful in the study of subsonic compressible flows is that the theory of

incompressible flows (i.e., the theory of analytic functions of a complex variable) is so well known.

The supersonic case is attacked by perfectly analogous means. As has been indicated by the author, there exist various operators for transforming solutions of one equation into solutions of another equation; and often for different purposes it is convenient to use different operators or different forms of the same operator. In the present report two operators which transform solutions of the relatively simple hyperbolic equation

$$\frac{\partial^2\psi}{\partial\Lambda^2} - \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{12}$$

(i.e., a pair of differentiable functions

$$\left.\begin{aligned} & g(\xi) \quad \text{and} \quad h(\eta) \\ & \text{where} \quad \xi = (\Lambda + \theta) \quad \text{and} \quad \eta = (\Lambda - \theta) \end{aligned}\right\} \tag{13}$$

of one variable) into solutions of the compressibility equations are derived.

As soon as the function $\psi(\Lambda,\theta)$ satisfying equation (9) is obtained, the corresponding flow in the physical plane can be obtained exactly in the same manner as in the subsonic case. (See reference 1, section 14.)

In contrast to the subsonic case, flows which in the logarithmic plane satisfy equation (12), Λ being some function of the speed, have not, to the knowledge of the author of this report, been studied.[6] Relatively little is known concerning how to choose g and h in order to obtain in the physical plane a flow around a prescribed obstacle, although there is reason to expect that this investigation should be simpler than in the subsonic case. However, this question is not considered herein, but is postponed to a future publication.

The following sections of this report are devoted exclusively to the determination of two different operators which transform differentiable functions of one real variable into solutions of equation (9).

[6] In reference 8 Coburn has indicated the extension of the Chaplygin–Kármán–Tsien method to the case of supersonic flows. The stream functions of such flows satisfy equation (9). Coburn indicated one characteristic property of these flows. On the other hand, the general properties and the form of flows (in the physical plane) which are of interest for the purpose of the present approach have not been investigated in Coburn's paper.

Operator Obtained by the Use of Riemann's Function

One method of generating stream functions of supersonic flows from expression (13) is by the use of Riemann's function.

As indicated in the preceding section, the stream function $\psi(\Lambda,\theta)$ satisfies equation (9). If, instead of Λ and θ, the variables ξ and η defined in expression (13) are introduced, and, instead of ψ, the "reduced stream function"

$$\psi^* = \psi \exp\left[-\int_a^{\xi+\eta} N_1(\tau)\ d\tau\right] \tag{14}$$

(where a is an arbitrary constant) is considered, then a formal computation shows that ψ^* satisfies the equation

$$\psi^*_{\xi\eta} + F_1(\xi + \eta)\psi^* = 0 \tag{15}$$

where

$$F_1 = -{N_1}^2 - \frac{dN_1}{d(2\Lambda)} = \frac{k+1}{64}\left[\frac{5(k+1)}{B^6} + \frac{12k}{B^4} + \frac{6k-14}{B^2} + (4k+8) - (3k-1)B^2\right] \tag{16}$$

where

$$B^2 = M^2 - 1 \tag{17}$$

Remark.– Obviously it is allowed to take for the lower limit a of the integral in equation (14) an arbitrary point, since replacing a by a^*, $a^* \neq a$, means only that ψ^* is multiplied by a constant, namely $\exp\left[\int_a^{a^*} N_1(\tau)\ d\tau\right]$. The most natural choice is to take for a the same value as in the subsonic case. (See formula (111) of reference 1.)

As is explained in more detail in appendix C, it is possible to extend the variable Λ to the complex values $l = \lambda + i\Lambda$. Then in the subsonic case the variable l assumes values on the negative real axis λ and on the positive imaginary axis Λ in the supersonic case. The integration can be then taken from $\tau = -\infty$. Since $N(0) = \infty$, the

integration must be performed along a path which avoids $l = 0$, for example, along the real axis from $-\infty$ to λ_1, $\lambda_1 < 0$; then along a circle with radius $|\lambda_1|$ and center at $l = 0$; and finally along a segment of the positive imaginary axis from $\tau = i|\lambda_1|$ to $\tau = i\Lambda$.

Let $R(\xi,\eta;\xi_o,\eta_o)$ be the Riemann's function of equation (15). (See, e.g., pp. 311–317 of reference 9.) Then, according to a classical result (p. 316, formula (7') of reference 9),

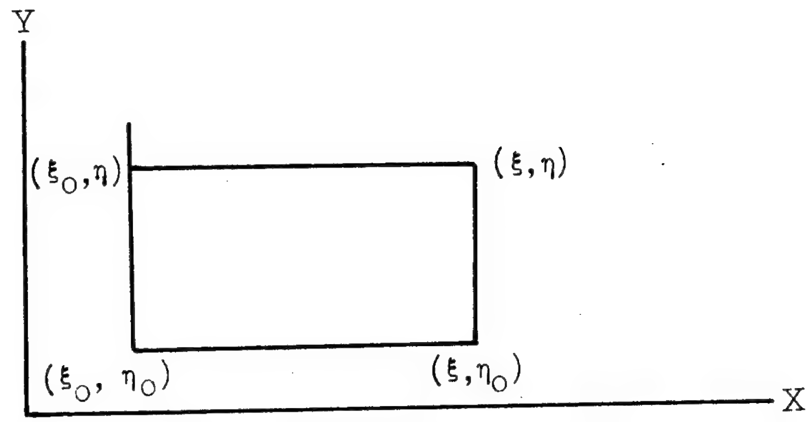
$$\begin{aligned}\psi^*(\xi,\eta) &= \psi^*(\xi_o,\eta_o)\, R(\xi,\eta;\xi_o,\eta_o) \\ &+ \int_{\eta_o}^{\eta} \psi^*_Y(\xi_o,Y)\, R(\xi_o,Y;\xi_o,\eta_o)\, dY \\ &+ \int_{\xi_o}^{\xi} \psi^*_X(X,\eta_o)\, R(X,\eta_o;\xi_o,\eta_o)\, dX \end{aligned} \tag{18}$$

where $\psi^*_X = \dfrac{\partial\psi^*}{\partial X}$, and so forth.

Since for $\psi^*(\xi_o,\eta_o)$ can be chosen an arbitrarily prescribed constant A and for $\psi^*(\xi_o,Y)$ and $\psi^*(X,\eta_o)$ can be chosen two arbitrary continuous functions $h(X)$ and $g(Y)$ $\big($provided that $h(\xi_o) = g(\eta_o)\big)$ the formula

$$\begin{aligned}\psi^*(\xi,\eta) &= AR(\xi,\eta;\xi_o,\eta_o) + \int_{\eta_o}^{\eta} h(Y)R(\xi_o,Y;\xi_o,\eta_o)\, dY \\ &+ \int_{\xi_o}^{\xi} g(X)R(X,\eta_o;\xi_o,\eta_o)\, dX \end{aligned} \tag{19}$$

represents an integral operator which transforms two arbitrary continuous functions of a real variable into a solution of equation (15).



The rectangle within which the values of $\psi^*(\xi,\eta)$ are determined by prescribing its values along the line segments: $Y = \eta_o$, $\xi_o \leq X \leq \xi$ and $X = \xi_o$, $\eta_o \leq Y \leq \eta$.

As is well known, the Riemann function of equation (15) can be written in the form

$$R(\xi,\eta;\xi_o,\eta_o) = \sum_{n=0}^{\infty} (-1)^n R^{(n)}(\xi,\eta;\xi_o,\eta_o) \tag{20}$$

where

$$\left.\begin{aligned}
R^{(o)}(\xi,\eta;\xi_o,\eta_o) &= 1 \\
R^{(1)}(\xi,\eta;\xi_o,\eta_o) &= \int_{\xi_o}^{\xi}\int_{\eta_o}^{\eta} F_1(X + Y)\,dX\,dY \\
R^{(2)}(\xi,\eta;\xi_o,\eta_o) &= \int_{\xi_o}^{\xi}\int_{\eta_o}^{\eta} F_1(X + Y)R^{(1)}(X,Y;\xi_o,\eta_o)\,dX\,dY \\
R^{(n)}(\xi,\eta;\xi_o,\eta_o) &= \int_{\xi_o}^{\xi}\int_{\eta_o}^{\eta} F_1(X + Y)R^{(n-1)}(X,Y;\xi_o,\eta_o)\,dX\,dY
\end{aligned}\right\} \tag{21}$$

(See reference 10.)

Except for the fact that $F_1(0) = \infty$, which causes certain complications, the evaluation of the foregoing integrals does not represent any essential difficulty. The tabulation of $R = R(\xi,\eta;\xi_o,\eta_o)$ is rather complicated since R is a function of four variables.

In appendix A the second term $R^{(1)}$ of the expansion (equation (20)) is computed and tabulated.

Remark.– In reference 1, formula (119)[7], the expression for a fundamental solution of equation (121) has been derived. The coefficient a of the logarithmic term is given by equation (123). It has exactly the same structure as the expression for $R(\xi,\eta;\xi_o,\eta_o)$, except for the fact that the arguments of a are complex quantities, whereas $\xi,\eta;\xi_o,\eta_o$ are real. In reference 11, Kraft has elaborated a working procedure for the evaluation of a which can be used for actual computation of R. It becomes in this case simpler since the arguments are real.

Integral Operator of Type Given by Formula (11)

In analogy to the subsonic case, where formula (11) yields a representation for solutions of equation (10) in terms of an arbitrary analytic function of one complex variable, there is derived in this section a representation distinct from the one explained in the preceding section for solutions of equation (9) (the equation for the stream function in the supersonic case) in terms of two arbitrary differentiable functions of a real variable.

Obviously, instead of equation (9), equation (15) for the reduced stream function ψ^* may be considered. The desired representation can be obtained as a consequence of the following theorem.

Theorem[8].– Let $E^{(n)}(\tau)$ denote functions defined by the following recurrence relations:

$$\left.\begin{aligned} E^{(1)}(\tau) &= -\int_a^\tau F_1(\tau_1)\,d\tau_1 \\ E^{(n+1)}(\tau) &= -\int_a^\tau \left[E^{(n)}{}_{\tau_1\tau_1}(\tau_1) + F_1(\tau_1)E^{(n)}(\tau_1)\right] d\tau_1 \end{aligned}\right\} \qquad (22)$$

---

[7]Note that in equation (119) and line 10 of page 46 of reference 1, by a misprint, a capital A instead of a, lower case, is used.

[8]This theorem has been indicated in reference 3; see chapters 4 and 5 of this reference. See also reference 12.

Here $F_1(\tau)$ is a given function, the derivatives of which satisfy the inequalities

$$\left|\frac{d^n F_1(\tau)}{d\tau^n}\right| \leqq \frac{(n+1)!M}{(\tau_o - \epsilon - \tau)^{n+2}}, \qquad n = 0,1,2,3, \ldots \tag{23}$$

where $\epsilon(>0)$, $M = M(\epsilon)$, and τ_o are conveniently chosen constants. Then the series

$$V^{(1)}(\xi,\eta) = f(\xi) + \sum_{n=1}^{\infty} E^{(n)}(\xi+\eta)f^{(n)}(\xi) \tag{24}$$

and

$$V^{(2)}(\xi,\eta) = g(\eta) + \sum_{n=1}^{\infty} E^{(n)}(\xi+\eta)g^{(n)}(\eta) \tag{25}$$

where f and g are two arbitrary differentiable functions of a real variable and

$$\left.\begin{aligned} f^{[n+1]}(\xi) &= \int_0^{\xi} f^{[n]}(\xi_1)\,d\xi_1 \\ g^{[n+1]}(\eta) &= \int_0^{\eta} g^{[n]}(\eta_1)\,d\eta_1 \end{aligned}\right\} \tag{26}$$

will converge in the intersection of the domains

$$\left[-1 < \frac{\xi}{\tau_o - \xi - \eta} < 1\right] \quad \text{and} \quad \left[-1 < \frac{\eta}{\tau_o - \xi - \eta} < 1\right] \tag{27}$$

and will represent there solutions of the equation

$$H(\psi^*) \equiv \psi^*_{\xi\eta} + F_1(\xi+\eta)\psi^* = 0 \tag{28}$$

The proof of this theorem is given in appendix B. As has been shown in reference 13 (see equation (45)), F_1 is given by equation (16), where M, B, and 2Λ are connected by relations (5) and (17). Thus the question of the representation of solutions of equation (15) is reduced to the investigation of the behavior of the function $F_1(\tau)$, in particular to determining whether its derivatives satisfy the inequalities (23). This investigation has been carried out in appendix C. If in the Λ,θ-plane the origin is shifted, thus introducing instead of 2Λ the new variable

$$2\Lambda' = 2\Lambda - \left(\frac{1}{h} - 1\right)\pi \tag{29}$$

and the variables are denoted by ξ' and η' as follows,

$$\left.\begin{aligned} \xi' &= (\Lambda' + \theta) \\ \eta' &= (\Lambda' - \theta) \end{aligned}\right\} \tag{30}$$

then equation (15) is transformed into an equation of the same form

$$\psi^*_{\xi'\eta'} + F_2(\xi' + \eta')\psi^* = 0 \tag{31}$$

where $F_2(\tau)$, $\tau = \xi' + \eta'$, satisfies the inequalities (23), for $F_2\left[2\Lambda + \pi\left(\frac{1}{h} - 1\right)\right] = F_1(2\Lambda)$ is an analytic function of Λ which is regular in the strip

$$-\infty < \lambda < \infty, \qquad 0 < \Lambda < \pi\left(h^{-1} - 1\right) \tag{32}$$

and therefore in particular in the circle of radius $\tau_o - \epsilon \equiv \frac{1}{2}\pi\left(\frac{1}{h} - 1\right) - \epsilon$, $\epsilon > 0$. (See fig. 2.) If $M = M(\epsilon)$ denotes the maximum of $F_1(2\Lambda)$ in this circle, then by classical results of the theory of functions of a complex variable, F_2 satisfies the inequalities (23), where $2\Lambda'$ is substituted for τ. Since ϵ can be chosen arbitrarily small, the series (equations (24) and (25)) formed for equation (31) converge in the domain indicated in figure 3.

On the other hand, in many instances it is necessary to have representations (equations (24) and (25)) of the solutions which are valid in the whole strip,

$$-\infty < \theta < \infty, \qquad 0 < 2\Lambda < 2\pi\left(\frac{1}{h} - 1\right) \tag{33}$$

One possibility of obtaining such formulas consists in approximating $F_2(2\Lambda')$ by a sequence of functions $F_2^{(m)}(2\Lambda')$, each of which is an entire function of Λ and therefore (by a classical result of the theory of analytic functions of a complex variable) satisfies the inequalities (23) and which is chosen in such a manner that

$$\lim_{m \longrightarrow \infty} F_2^{(m)}(2\Lambda') = F_2(2\Lambda') \tag{34}$$

in the interval

$$0 < 2\Lambda < 2\left(\frac{1}{h} - 1\right)\pi \tag{35}$$

(Concerning another method of representing $V^{(k)}$, $k = 1,2$, see (27), (25) in the domain (33), and reference 2, section 5.) Let $\psi^{*(m)}(\xi',\eta')$ denote the solutions of the equation

$$\psi^{*(m)}{}_{\xi'\eta'} + F_2^{(m)}(\xi' + \eta')\psi^{*(m)} = 0, \quad m = 1,2, \ldots \tag{36}$$

where

$$-\xi^{(0)} < \xi' < \xi^{(0)}$$

such that $\psi^{*(m)}(\xi',0) = \psi^{*}(\xi',0)$ and $\psi^{*(m)}(0,\eta') = \psi^{*}(0,\eta')$, $-\eta^{(0)} < \eta' < \eta^{(0)}$. As is shown in appendix E,

$$\lim_{m \longrightarrow \infty} \psi^{*(m)}(\xi',\eta') = \psi^{*}(\xi',\eta') \tag{37}$$

in the interval

$$-\xi^{(0)} < \xi' < \xi^{(0)}, \qquad -\eta^{(0)} < \eta' < \eta^{(0)} \tag{38}$$

A sequence of functions $F_2^{(m)}(2\Lambda')$ can be obtained in the following manner. Let

$$F_2(2\Lambda') = \alpha_0 + \alpha_1\Lambda' + \alpha_2\Lambda'^2 + \ldots \tag{39}$$

be the series development of F_2 around the point $2\Lambda' = 0$ $\left(\text{i.e.,}\quad 2\Lambda = \left(\frac{1}{h} - 1\right)\pi\right)$. The series $\sum_{n=0}^{\infty} \alpha_n \Lambda'^n$ obviously converges in the circle indicated in figure 2. The series

$$F_2^{(m)}(2\Lambda') = \sum_{n=0}^{\infty} \frac{\alpha_n}{\Gamma(1 + n/m)} \Lambda'^n \tag{40}$$

will be, for every finite positive number m, an entire function of Λ' and therefore the series (equations (24) and (25)) obtained for equation (36) with $F_2^{(m)}$ given by equation (40) will converge, for any $m < \infty$, in the strip (33). On the other hand, according to a classical theorem of the theory of functions (see reference 14), the relation (equation (34)) holds in this case, so that for the functions $\psi^{*(m)}(\xi',\eta')$ obtained in the foregoing manner the relation (equation (37)) holds in the whole interval (38).

Naturally it is also possible to approximate $F_2(2\Lambda')$ by using different procedures. Some of them are convenient for some special purposes, in particular if a problem requires the representation of solutions in certain subdomains of the strip (33).

SUMMARY REMARKS

The present paper employs the hodograph method for generating flow patterns of supersonic flows. The essential feature of this method is that the equations are linear and therefore the principle of superposition of solutions holds. If, therefore, $\psi_\nu(\Lambda,\theta)$, $\nu = 1,2,3, \ldots$, represents a set of particular solutions of equation (7), and A_ν, arbitrary constants, any linear combination

$$\sum_{\nu=1}^{N} A_\nu \psi_\nu(\Lambda,\theta)$$

is also a solution of equation (7). By varying the constants A_ν, $\nu = 1,2, \ldots$, flows around different shapes can be obtained. On the other hand, it is often necessary to determine constants A_ν to yield a flow which approximates that about a prescribed boundary curve, the

equation of which is, say, $F(x,y) = 0$. In this case, the constants A_ν can be determined in such a manner that

$$M(A) \equiv \int \left| F\left[x(\Lambda,\theta;A_\nu),y(\Lambda,\theta;A_\nu)\right]\right|^2 d\theta$$

under the condition

$$\psi(\Lambda,\theta) = 0$$

Here $x = x(\Lambda,\theta;A_\nu)$ and $y = y(\Lambda,\theta;A_\nu)$ represent functions of Λ and θ defined in the INTRODUCTION and corresponding to the stream function $\psi = \sum A_\nu\psi_\nu(\Lambda,\theta)$ in the Λ,θ-plane.

As is well known, a solution of the problem does not always exist. Under the assumption that $\psi_\nu(\Lambda,\theta)$, $\nu = 1,2, \ldots$, represents a system which is in a certain sense complete, $M(A)$ represents a measure for closeness of approximation. In this case the fact that $M(A)$ approaches zero when the number N of functions $\psi_\nu(x,y)$ increases can be considered as a condition for the existence of a solution.

On the other hand, in many instances, only solutions of the problem exist which possess shock lines, and therefore not only the usual condition that one of the streamlines coincides with the given boundary must be considered but also the condition that these shock conditions are satisfied along <u>unknown</u> characteristics.

Brown University
Providence, R.I., November 18, 1946

APPENDIX A

EVALUATION AND TABULATION OF THE FUNCTION

$R^{(1)}(\xi,\eta;\xi_o,\eta_o)$

The second term $R^{(1)}(\xi,\eta;\xi_o,\eta_o)$ in the series for the Riemann function is evaluated in this section.

F_1 is a function of the variable $\Lambda = \frac{\xi + \eta}{2}$ and therefore

$$\int_{\xi_o}^{\xi} F_1(X + \eta)\, dX = \Gamma^{(1)}(\xi + \eta) - \Gamma^{(1)}(\xi_o + \eta)$$

$$\int_{\eta_o}^{\eta} dY \int_{\xi_o}^{\xi} F(X + Y)\, dX = \int_{\eta_o}^{\eta} \left[\Gamma^{(1)}(\xi + Y) - \Gamma^{(1)}(\xi_o + Y)\right] dY$$

$$= \Gamma^{(2)}(\xi + \eta) - \Gamma^{(2)}(\xi + \eta_o) - \Gamma^{(2)}(\xi_o + \eta) + \Gamma^{(2)}(\xi_o + \eta_o) \qquad (41)$$

where

$$\int_{t_o}^{t_1} F_1(\tau)\, d\tau = \Gamma^{(1)}(t_1) - \Gamma^{(1)}(t_o) \qquad (42)$$

and

$$\int_{t_o}^{t_1} \Gamma^{(1)}(\tau)\, d\tau = \Gamma^{(2)}(t_1) - \Gamma^{(2)}(t_o) \qquad (43)$$

The fact that $F(0)$ is infinite causes a certain amount of inconvenience in the tabulation of the functions $\Gamma^{(1)}$ and $\Gamma^{(2)}$. According to equation (16), for $k = 1.4$,

$$F_1(B) = \frac{0.45}{B^6} + \frac{0.63}{B^4} - \frac{0.21}{B^2} - 0.51 - 0.12B^2 \tag{44}$$

where

$$\Lambda = 2\sqrt{6} \text{ arc tan } \frac{B}{\sqrt{6}} - 2 \text{ arc tan } B \tag{45}$$

and B^2 is given by equation (17), M being the Mach number.

In table I the values of M, B, and F_1 have been tabulated; in figure 4, F_1 is given as a function of M, B, and 2Λ. For $2\Lambda = 0.3686$ the function F_1 vanishes, that is, $F_1\left[B(0.3686)\right] = 0$; this point has been chosen as the initial point for the integration, since, as has been pointed out earlier, for the most natural choice $2\Lambda = 0$, F_1 becomes infinite.

The values of $\Gamma^{(1)}(2\Lambda)$ and $\Gamma^{(2)}(2\Lambda)$ for $\Gamma^{(1)}(0.3686) = \Gamma^{(2)}(0.3686) = 0$ are given in table II. The interval of tabulation of 2Λ is one four-hundredth. In order to obtain $\Gamma^{(2)}(2\Lambda)$ for intermediate values, the interpolation formula

$$\Gamma^{(2)}(2\Lambda) = \Gamma^{(2)}(2\Lambda_o) + pD_1 + p^2D_2 \tag{46}$$

can be used, where

$$\Gamma^{(2)}(2\Lambda) = \Gamma^{(2)}(2\Lambda_o + ph) = \Gamma^{(2)}(2\Lambda_o) + ph + \frac{1}{2}p(p - 1)h^2$$

$$= \Gamma^{(2)}(2\Lambda_o) + pD_1 + p^2D_2 \tag{47}$$

The quantity h is the interval of tabulation at $2\Lambda_o$.

APPENDIX B

PROOF OF THEOREM GIVEN BY EQUATIONS (22) TO (28)

The proof is carried out at first under the assumption that the series[9] (equations (24) and (25)) as well as those for their first derivatives converge uniformly. (The validity of this assumption is proved in this appendix.) A formal computation yields

$$V^{(1)}{}_{\xi'\eta'} = fE^{(1)}{}_{\eta'} + \sum_{n=1}^{\infty} f^{[n]}\left[E^{(n)}{}_{\xi'\eta'} + E^{(n+1)}{}_{\eta'}\right] \tag{48}$$

$$F_1V^{(1)} = fF_1 + \sum_{n=1}^{\infty} f^{[n]}F_1E^{(n)} \tag{49}$$

that is,

$$V^{(1)}{}_{\xi'\eta'} + F_1V^{(1)} = f\left[E^{(1)}{}_{\eta'} + F_1\right] + \sum_{n=1}^{\infty} f^{[n]}\left[E^{(n)}{}_{\xi'\eta'} + E^{(n+1)}{}_{\eta'} + F_1E^{(n)}\right] \tag{50}$$

where F_1 is a function of one variable $\tau = 2\Lambda'$.

Assuming that the $E^{(n)}$ functions are also functions of one variable τ,

$$\left.\begin{aligned} E^{(n)}{}_{\eta'} &= E^{(n)}{}_{\tau} \\ E^{(n)}{}_{\xi'\eta'} &= E^{(n)}{}_{\tau\tau} \end{aligned}\right\} \tag{51}$$

which yields the foregoing relations, if the assumption is made that

$$E^{(n)}(a) = 0 \tag{52}$$

[9]In this appendix Λ,ξ,η are replaced by Λ',ξ',η'.

It remains therefore to prove that the series and both its formal (i.e., term-by-term) derivatives converge uniformly.

Notation.- If in the interval I there hold for the functions $A(\tau)$ and $\tilde{A}(\tau)$ and all their derivatives the inequalities:

$$\left|A(\tau)\right| < \tilde{A}(\tau) \quad \text{and} \quad \left|\frac{d^n A(\tau)}{d\tau^n}\right| \leqq \frac{d^n \tilde{A}(\tau)}{d\tau^n}, \quad \tau \epsilon I \tag{53}$$

$\tilde{A}(\tau)$ will be denoted as a dominant of $A(\tau)$, which fact is symbolized by writing $A \ll \tilde{A}$ or

$$\tilde{A} \gg A, \quad \tau \epsilon I \tag{54}$$

If $\tilde{E}^{(1)}(\tau)$ is given by

$$\tilde{E}^{(1)}(\tau) = \int_0^\tau \tilde{F}_1(\tau_1)\, d\tau_1 \tag{55}$$

where

$$\tilde{F}_1(\tau) \gg F_1(\tau) \tag{56}$$

then

$$\left|E^{(1)}(\tau)\right| = \left|\int_a^\tau F_1(\tau_1)\, d\tau_1\right| \leqq \int_a^\tau \tilde{F}_1(\tau_1)\, d\tau \leqq \tilde{E}^{(1)}(\tau), \quad \tau \epsilon I \tag{57}$$

and also

$$\left|\frac{d^n E^{(1)}(\tau)}{d\tau^n}\right| \leq \frac{d^n \tilde{E}^{(1)}(\tau)}{d\tau^n}, \quad \tau \epsilon I \tag{58}$$

Thus

$$E^{(1)}(\tau) \ll \tilde{E}^{(1)}(\tau), \quad \tau \epsilon I \tag{59}$$

Suppose now that

$$\tilde{E}^{(n+1)}(\tau) = \int_a^{\tau} \left[\tilde{E}^{(n)}{}_{\tau_1\tau_1}(\tau_1) + \tilde{F}_1(\tau_1)\tilde{E}^{(n)}(\tau_1)\right] d\tau_1 + \tilde{H}^{(n)}(\tau) \tag{60}$$

where

$$E^{(n)}(\tau) \ll \tilde{E}^{(n)}(\tau), \qquad 0 \ll \tilde{H}^{(n)}(\tau), \ \tau\epsilon I \tag{61}$$

Then it follows immediately that

$$\left.\begin{aligned} \left|E^{(n+1)}(\tau)\right| &\leq \tilde{E}^{(n+1)}(\tau) \\ \left|\frac{dE^{(n+1)}(\tau)}{d\tau}\right| &\leq \frac{d\tilde{E}^{(n+1)}(\tau)}{d\tau} \end{aligned}\right\} \tag{62}$$

and, by considering the corresponding derivatives of $\left[\tilde{E}^{(n)}{}_{\tau\tau} + \tilde{F}_1\tilde{E}^{(n)}\right]$ in comparison with $\left[E^{(n)}{}_{\tau\tau} + F_1E^{(n)}\right]$, it follows that

$$E^{(n+1)}(\tau) \ll \tilde{E}^{(n+1)}(\tau), \ \tau\epsilon I \tag{63}$$

which completes the proof by induction.

Now by expression (23),

$$F_1(\tau) \ll \tilde{F}_1(\tau) = \frac{M}{(\tau_o - \epsilon - \tau)^{2n}}, \ \tau\epsilon I \tag{64}$$

If $\tilde{H}^{(n)}(\tau)$ is given by

$$\tilde{H}^{(n)} = \frac{C_n M \tau_o{}^2}{(\tau_o - \epsilon - a)^n} \tag{65}$$

where the C_n's are some conveniently chosen positive constants (to be determined later), then an explicit expression may be obtained for $\tilde{E}(\tau)$, namely,

$$\tilde{E}^{(n)}(\tau) = \frac{C_n M \tau_o^2}{(\tau_o - \epsilon - \tau)^n} \tag{66}$$

In order to express C_{n+1} in terms of C_n, the right-hand side of equation (66) is substituted into equation (60) to give

$$\tilde{E}^{(n+1)}(\tau) = C_n M \tau_o^2 \int_a^\tau \left[\frac{n(n+1) + C_n M \tau_o^2}{(\tau_o - \epsilon - \tau)^{n+2}}\right] d\tau + \frac{C_{n+1} M \tau_o^2}{(\tau_o - \epsilon - a)^n} = \frac{C_{n+1} M \tau_o^2}{(\tau_o - \epsilon - \tau)^{n+2}} \tag{67}$$

where

$$C_{n+1} = C_n \left(n + \frac{M \tau_o^2}{n + 1}\right) \tag{68}$$

Thus for sufficiently large values of n

$$C_{n+1} \leqq C_n (n + 1) \tag{69}$$

and therefore for every value of M

$$C_{n+1} \leqq M^* (n + 1)! \tag{70}$$

where M^* is a conveniently chosen constant. The function $f(\xi)$ is assumed to be differentiable and therefore there exists a constant, say, M_2, such that

$$|f(\xi')| < M_2 \quad \text{for} \quad |\xi'| \leqq \xi_2 \tag{71}$$

Then

$$\left|f^{[n]}(\xi')\right| \leqq \frac{M_2 \xi'^n}{n!} \quad \text{for} \quad |\xi'| \leqq \xi_2 \tag{72}$$

Consequently, from equation (24) it is seen that

$$V_1(\xi',\eta') \ll 2M^*M_2\Lambda_o{}^2M\left[1+\frac{|\xi'|2!}{(\tau_o-\epsilon-\tau)}+\ldots\frac{|\xi'|^n(n+1)!}{n!(\tau_o-\epsilon-\tau)^n}+\ldots\right] \tag{73}$$

which converges if

$$-1 < \frac{\xi'}{(\tau_o - \epsilon - \tau)} \equiv \frac{\Lambda' + \theta}{(\tau_o - \epsilon - 2\Lambda')} < 1 \tag{74}$$

Since ϵ can be chosen arbitrarily small, the series (equation (24)) converges in the domain

$$-1 < \frac{\Lambda' + \theta}{\tau_o - 2\Lambda'} < 1 \tag{75}$$

(see fig. 3), where the domain (75) is bounded by solid lines. Similarly, the series (equation (25)) converges in the domain

$$-1 < \frac{\Lambda' - \theta}{\tau_o - 2\Lambda} < 1 \tag{76}$$

which is bounded by a dashed line so that both series (equations (24) and (25)) converge in the shaded domain.

APPENDIX C

INVESTIGATION OF THE FUNCTION $F_1(2\Lambda)$

This section is devoted to the investigation of the behavior of the function $F_1(2\Lambda)$ in order to determine where its derivatives satisfy the inequalities (64).

If M varies from 1 to ∞, $B = \left(M^2 - 1\right)^{1/2}$ varies from 0 to ∞ and

$$\Lambda = \frac{1}{h} \text{ arc tan } (hB) - \text{arc tan } B \tag{5}$$

from 0 to $\left(\frac{1}{h} - 1\right)\frac{\pi}{2}$. Therefore it is necessary to investigate the behavior of $F_1(2\Lambda)$ on the interval

$$I = E\left[0 \leqq 2\Lambda \leqq \left(\frac{1}{h} - 1\right)\pi\right] \tag{77}$$

Since F_1 (given by equation (16)) is a rational function of B, it is sufficient to investigate the function $B(2\lambda)$.

In order to carry out this investigation it is convenient to continue the functions to complex values, so that it will be possible to use methods of the theory of analytic functions of a complex variable.

Instead of equation (5), the complex function

$$l(\tilde{\beta}) = \lambda + i\Lambda = \frac{1}{2}\left(\log \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} + \frac{1}{h} \log \frac{h^{-1} + \tilde{\beta}}{h^{-1} - \tilde{\beta}}\right) \tag{78}$$

of the complex variable $\tilde{\beta} = T + iB$ will be considered $\left(\tilde{\beta} = T,\ 0 \leq T < 1\right)$.

When $\tilde{\beta}$ is real it is seen that, if the principal values of the logarithmic terms are chosen, then l is also real and the foregoing equation reduces to:

$$\lambda = \frac{1}{2}\left(\log\frac{1 - T}{1 + T} + \frac{1}{h}\log\frac{h^{-1} + T}{h^{-1} - T}\right), \ \Lambda = 0 \tag{79}$$

in agreement with the earlier definition of λ. On the other hand, if

$$\tilde{\beta} = iB, \ B > 0$$

B is purely imaginary; then, if the principal values of the logarithmic terms are taken once again, there results:

$$\lambda = 0, \qquad i\Lambda = i\left[-\arc\tan B + \frac{1}{h}\arc\tan(hB)\right] \tag{80}$$

which is also in agreement with the previous definition. The question which has to be investigated is to determine the domain of regularity of $\beta(l)$, inverse to equation (77).

A classical theorem concerning the inversion of analytic functions states that, if $w(z)$ is a function of the complex variable z which is regular at $z = z_o$ and has a nonzero derivative there, then, in some sufficiently small neighborhood of the point $w = w_o = w(z_o)$, it is possible to invert the function $w(z)$, that is, to express z as a regular function of w. (See reference 15, p. 142.) At every point except $\beta = \pm 1$ and $\beta = \pm h$, the function $l(\tilde{\beta})$ is regular, so that the zero points of the derivative $dl/d\beta$ have to be determined.

A formal computation yields

$$\frac{dl}{d\tilde{\beta}} = \frac{1}{2}\left[-\frac{1}{1 - \tilde{\beta}} - \frac{1}{1 + \tilde{\beta}} + \frac{1}{h}\left(\frac{1}{h^{-1} + \tilde{\beta}} + \frac{1}{h^{-1} - \tilde{\beta}}\right)\right]$$

$$= -\frac{\left(1 - h^2\right)\tilde{\beta}^2}{\left(1 - \tilde{\beta}^2\right)\left(1 - h^2\tilde{\beta}^2\right)} \tag{81}$$

Obviously,

$$\frac{dl}{d\tilde{\beta}} = 0 \quad \text{for} \quad \tilde{\beta} = 0 \quad \text{and} \quad \tilde{\beta} = \infty \tag{82}$$

The discussion would become quite complicated if all the possible branches of the function $l(\tilde{\beta})$ and of the inverse $\tilde{\beta}(l)$ were to be considered, because each summand of equation (81) is an infinitely

many-valued function of $\tilde{\beta}$, having logarithmic singularities at ± 1 and $\pm h^{-1}$, respectively. It is therefore convenient to render $l(\tilde{\beta})$ single-valued by the artifice of starting at $\tilde{\beta} = 0$ with the principal values of $\log(1 - \tilde{\beta})$, $\log(1 + \tilde{\beta})$, $\log(h^{-1} + \tilde{\beta})$ and $\log(h^{-1} - \tilde{\beta})$ (i.e., with the real values of the logarithms) and slitting the complex $\tilde{\beta}$-plane from 1 to $+\infty$ and from -1 to $-\infty$ along the real axis. Since $h^{-1} > 1$, it is clear that the slit plane contains no branch points and therefore l is uniquely defined by beginning with the aforementioned determination of $l(0)$.

Now, a direct investigation shows that the $\tilde{\beta}$-plane, slit in the manner just described, is mapped into the band

$$-\infty < \lambda < \infty, \ -(h^{-1} - 1)\pi < \Lambda < (h^{-1} - 1)\pi \tag{83}$$

However, the mapping is not bi-unique, for at the branch point $\tilde{\beta} = 0$ the mapping is easily seen to be 3 to 1. (See appendix D.) Thus, the simply covered slit $\tilde{\beta}$-plane is mapped into the simply covered band just defined. The only singularities of $\tilde{\beta}(l)$ in the strip

$$-\infty < \lambda < \infty, \qquad 0 \leq \Lambda \leq (h^{-1} - 1)\pi \tag{84}$$

are the points $l = 0$ where $\tilde{\beta}(l)$ has a branch point and $l = (h^{-1} - 1)\pi i$ where $\tilde{\beta}(l)$ has a pole.

APPENDIX D

THE DEVELOPMENT OF $\tilde{\beta}(l)$ AT THE ORIGIN $l = 0$

The simplest method of proving that $l = 0$ is a branch point of $\tilde{\beta}(l)$ of third order is to determine the series development of this function at $l = 0$. In order to obtain this development, the principal branch of $l(\tilde{\beta})$, namely,

$$\frac{1}{2}\left[\log\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right) + \frac{1}{h}\log\left(\frac{h^{-1}+\tilde{\beta}}{h^{-1}-\tilde{\beta}}\right)\right] = i\left[-\arc\tan\left(\tilde{\beta}/i\right) + \frac{1}{h}\arc\tan\left(\tilde{\beta}h/i\right)\right] \quad (85)$$

has to be developed in a power series to give

$$l = -\frac{\tilde{\beta}^3}{3}\left(1-h^2\right) - \frac{\tilde{\beta}^5}{5}\left(1-h^4\right) - \frac{\tilde{\beta}^7}{7}\left(1-h^6\right)\ .\ .\ . \quad (86)$$

or

$$-\frac{3}{1-h^2}\,l = \tilde{\beta}^3 + \frac{3}{5}\frac{\left(1-h^4\right)}{\left(1-h^2\right)}\tilde{\beta}^5 + \frac{3}{7}\frac{\left(1-h^6\right)}{\left(1-h^2\right)}\tilde{\beta}^7 + \ .\ .\ . \quad (87)$$

Introducing a new variable s,

$$s^3 = -\frac{3}{1-h^2}\,l, \quad \text{that is,} \quad s = \left(-\frac{3}{1-h^2}\,l\right)^{1/3} \quad (88)$$

yields the development

$$s = c_1\tilde{\beta} + c_3\tilde{\beta}^3 + c_5\tilde{\beta}^5 + \ .\ .\ . \quad (89)$$

Since c_1, as is easily seen, is not zero, it follows from the general theory of analytic functions that this series can be inverted:

$$\tilde{\beta} = d_1 s + d_2 s^3 + d_3 s^5 + \ .\ .\ . \quad (90)$$

In particular, for $h = 1/\sqrt{6}$,

$$c_1 = 1, \quad c_3 = 0.2333, \ . \ . \ . \qquad d_1 = 1, \quad d_3 = -0.2333 \ . \ . \ . \tag{91}$$[10]

[10]The values of c_n and d_n, $n = 1, 3, \ldots, 25$ have been computed by the author and will be published elsewhere.

APPENDIX E

APPROXIMATIONS OF SOLUTIONS

In the present report, solutions of equation (15) have been approximated at several places by solutions of an equation in which the coefficient F_1 is replaced by another coefficient which approximates it. The legitimacy of such a procedure is established by the following theorem.

Theorem.– Let $\psi^{(k)}(\xi,\eta)$, $k = 1,2$, be solutions of the equations

$$\psi_{\xi\eta}{}^{(k)} + f^{(k)}\psi^{(k)} = 0 \tag{92}$$

where the $f^{(k)}$'s are given functions of ξ and η which satisfy throughout the rectangle $0 \leq \xi \leq a$, $0 \leq \eta \leq b$ the inequalities

$$\left| f^{(1)} - f^{(2)} \right| \leq \epsilon \quad \text{and} \quad \left| f^{(k)} \right| \leq m \tag{93}$$

ϵ and m being positive constants. The functions $\psi^{(k)}$ are also to satisfy the conditions:

$$\psi^{(1)}(\xi,0) = \psi^{(2)}(\xi,0) = X_1(\xi) \quad \text{for} \quad 0 \leq \xi \leq a \tag{94}$$

and

$$\psi^{(1)}(0,\eta) = \psi^{(2)}(0,\eta) = X_2(\eta) \quad \text{for} \quad 0 \leq \eta \leq b \tag{95}$$

where X_1 and X_2 are continuous functions prescribed on the closed intervals $(0,a)$ and $(0,b)$, respectively, subject only to the condition that $X_1(0) = X_2(0)$. Let α be a positive number such that $\left|X_1\right| \leq \alpha$, $\left|X_2\right| \leq \alpha$. Then there exists a positive number M, depending only on ϵ, α, m, and ab (the area of the given rectangle) such that for fixed values of α, m, and ab, M approaches a finite (positive) limit as ϵ approaches zero, and such that, throughout the given rectangle, the following inequality holds:

$$\left| \psi^{(1)}(\xi,\eta) - \psi^{(2)}(\xi,\eta) \right| \leq \epsilon M \tag{96}$$

Proof.– According to the general theory of linear hyperbolic partial differential equations, it is known that the functions $\psi^{(k)}$ satisfying expressions (92), (93), and (95) may be expressed in the form of the following infinite series, which is valid throughout the given rectangle:

$$\psi^{(k)}(\xi,\eta) = p(\xi,\eta) - \int_0^{\xi} \int_0^{\eta} f^{(k)}(\xi_1,\eta_1)p(\xi_1,\eta_1)\, d\xi_1\, d\eta_1$$

$$+ \int_0^{\xi} \int_0^{\eta} f^{(k)}(\xi_1,\eta_1)\left[\int_0^{\xi_1} \int_0^{\eta_1} f^{(k)}(\xi_2,\eta_2)p(\xi_2,\eta_2)\, d\xi_2\, d\eta_2\right] d\xi_1\, d\eta_1 \ldots \tag{97}$$

where

$$p(\xi,\eta) = X_1(\xi) + X_2(\eta) - X_1(0) \tag{98}$$

Let

$$\delta(\xi,\eta) = f^{(1)}(\xi,\eta) - f^{(2)}(\xi,\eta) \tag{99}$$

so that, according to expression (93), $\left|\delta(\xi,\eta)\right| \leq \epsilon$. Now set k equal to 1 and 2, respectively, in equation (97) and subtract the two equations so obtained (pairing off corresponding terms and replacing $f^{(1)}$ by $f^{(2)} + \delta$). The result obtained is:

$$
\begin{aligned}
\psi^{(1)} - \psi^{(2)} = &- \int_0^{\xi} \int_0^{\eta} \delta(\xi_1,\eta_1) p(\xi_1,\eta_1)\, d\xi_1\, d\eta_1 + \int_0^{\xi} \int_0^{\eta} \Bigg[f^{(2)}(\xi_1,\eta_1) \\
&+ \delta(\xi_1,\eta_1) \Bigg] \left\{ \int_0^{\xi_1} \int_0^{\eta_1} \left[f^{(2)}(\xi_2,\eta_2) + \delta(\xi_2,\eta_2) \right] p(\xi_2,\eta_2)\, d\xi_2\, d\eta_2 \right\} d\xi_1\, d\eta_1 \\
&- \int_0^{\xi} \int_0^{\eta} f^{(2)}(\xi_1,\eta_1) \left[\int_0^{\xi_1} \int_0^{\eta_1} f^{(2)}(\xi_2,\eta_2) p(\xi_2,\eta_2)\, d\xi_2\, d\eta_2 \right] d\xi_1\, d\eta_1
\end{aligned}
\tag{100}
$$

It is easily seen that the second term of the right-hand side of equation (100) can be rewritten in the form

$$
\begin{aligned}
&\int_0^{\xi} \int_0^{\eta} f^{(2)}(\xi_1,\eta_1) \left[\int_0^{\xi_1} \int_0^{\eta_1} \delta(\xi_2,\eta_2) p(\xi_2,\eta_2)\, d\xi_2\, d\eta_2 \right] d\xi_1\, d\eta_1 \\
&+ \int_0^{\xi} \int_0^{\eta} \delta(\xi_1,\eta_1) \left[\int_0^{\xi_1} \int_0^{\eta_1} f^{(2)}(\xi_2,\eta_2) p(\xi_2,\eta_2)\, d\xi_2\, d\eta_2 \right] d\xi_1\, d\eta_1 \\
&+ \int_0^{\xi} \int_0^{\eta} \delta(\xi_1,\eta_1) \left[\int_0^{\xi_1} \int_0^{\eta_1} \delta(\xi_2,\eta_2) p(\xi_2,\eta_2)\, d\xi_2\, d\eta_2 \right] d\xi_1\, d\eta_1
\end{aligned}
\tag{101}
$$

and it is clear how a similar reduction can be effected in all the succeeding terms, so that no sums appear in any of the integrands.

Now, if each term in the series on the right-hand side is replaced by its absolute value, an upper bound is obtained for the absolute value of the left-hand side of equation (100); and this inequality continues to hold if the functions p, δ, and $f^{(2)}$ are replaced by their respective upper bounds 3α, ϵ, and m. (The upper bound for p follows directly from equation (98) and the definition of α.) Therefore, from equation (100) there is obtained the inequality:

$$\left|\psi^{(1)} - \psi^{(2)}\right| \leq 3\alpha\epsilon\xi\eta + \frac{3\alpha\left(\epsilon^2 + 2\epsilon m\right)\xi^2\eta^2}{(2!)^2} + \frac{3\alpha\left(\epsilon^3 + 3\epsilon^2 m + 3\epsilon m^2\right)\xi^3\eta^3}{(3!)^2} + \ldots \tag{102}$$

It is not difficult to see that the nth term of the right-hand side of equation (102) is simply:

$$\frac{3\alpha\left[(\epsilon + m)^n - m\right]\xi^n\eta^n}{(n!)^2} \tag{103}$$

The inequality (102) holds _a fortiori_ if ξ and η are replaced by their respective upper bounds a and b. Thus, there is finally obtained for all points in the given rectangle, the inequality

$$\left|\psi^{(1)} - \psi^{(2)}\right| \leq \sum_{n=1}^{\infty} \frac{3\alpha\left[(\epsilon + m)^n - m^n\right](ab)^n}{(n!)^2} \tag{104}$$

This series is easily shown, by elementary tests, to converge. Furthermore, it is clear that the quantity ϵ can be factored out of the expression $\left[(\epsilon + m)^n - m^n\right]$, so that the series may be written as the product of ϵ by a new series, which is again easily shown to converge for all values of ϵ. Designating this new series by M, the conclusion of the theorem is obtained.

In less precise language than that used in the formulation of the theorem, the result obtained may be expressed by saying that, if the coefficients $f^{(k)}$ of the two differential equations (equation (92)) approximate each other very closely in a certain region and if the respective solutions satisfy the same side conditions (equations (94) and (95)), then the solutions approximate each other very closely throughout the given region.

In particular it follows from this theorem that, if the sequence of coefficients of the given hyperbolic equation converges uniformly, the corresponding sequence of solutions also converges uniformly to the solution of the limiting equation.

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TABLE I.– F_1 AS A FUNCTION OF M, B, AND 2Λ

M	B	2Λ	F_1
1.00	0.0000	0.0000	∞
1.04	.2857	.0120	
1.08	.4079	.0338	
1.12	.5044	.0606	
1.16	.5879	.0910	15.0172
1.20	.6633	.1242	7.4967
1.24	.7332	.1595	4.1109
1.28	.7990	.1964	2.3598
1.32	.8616	.2346	1.3609
1.36	.9217	.2738	.7475
1.40	.9798	.3137	.3483
1.44	1.0361	.3542	.0758
1.48	1.0911	.3950	–.1179
1.52	1.1447	.4362	–.2606
1.56	1.1973	.4774	–.3692
1.60	1.2490	.5187	–.4544
1.64	1.2998	.5600	–.5231
1.68	1.3500	.6012	–.5799
1.72	1.3994	.6422	–.6281
1.76	1.4483	.6829	–.6699
1.80	1.4967	.7234	–.7070
1.84	1.5445	.7637	–.7405
1.88	1.5920	.8035	–.7713
1.92	1.6390	.8430	–.8000
1.96	1.6857	.8821	–.8273
2.00	1.7321	.9208	–.8533

TABLE II.– F_1, $\Gamma^{(1)}$, AND $\Gamma^{(2)}$ AS FUNCTIONS OF 2Λ, M, AND B

2Λ	M	B	F_1	$\Gamma^{(1)}$	$\Gamma^{(2)}$
0.00	1.0000	0	∞	$-\infty$	$-\infty$
.04	1.0899	.4334	84.078	−2.6449	−0.1211
.08	1.1460	.5597	19.833	−.9974	−.0561
.12	1.1951	.6544	8.1146	−.4859	−.0279
.16	1.2406	.7341	4.0787	−.2537	−.0136
.20	1.2838	.8051	2.2408	−.1315	−.0061
.24	1.3255	.8701	1.2579	−.0634	−.0024
.28	1.3663	.9310	.6737	−.0257	−.0007
.32	1.4063	.9887	.2989	−.0067	−.0001
.36	1.4457	1.0441	.0441	−.0002	.0000
.40	1.4848	1.0976	−.1375	−.0022	−.0000
.45	1.5334	1.1625	−.3003	−.0134	−.0004
.50	1.5819	1.2257	−.4182	−.0315	−.0015
.55	1.6303	1.2876	−.5077	−.0548	−.0036
.60	1.6789	1.3486	−.5784	−.0820	−.0070
.65	1.7277	1.4089	−.6365	−.1124	−.0119
.70	1.7768	1.4687	−.6860	−.1455	−.0183
.75	1.8264	1.5283	−.7294	−.1809	−.0265
.80	1.8765	1.5878	−.7686	−.2184	−.0364
.85	1.9271	1.6474	−.8050	−.2577	−.0483
.90	1.9784	1.7071	−.8394	−.2988	−.0622

NACA

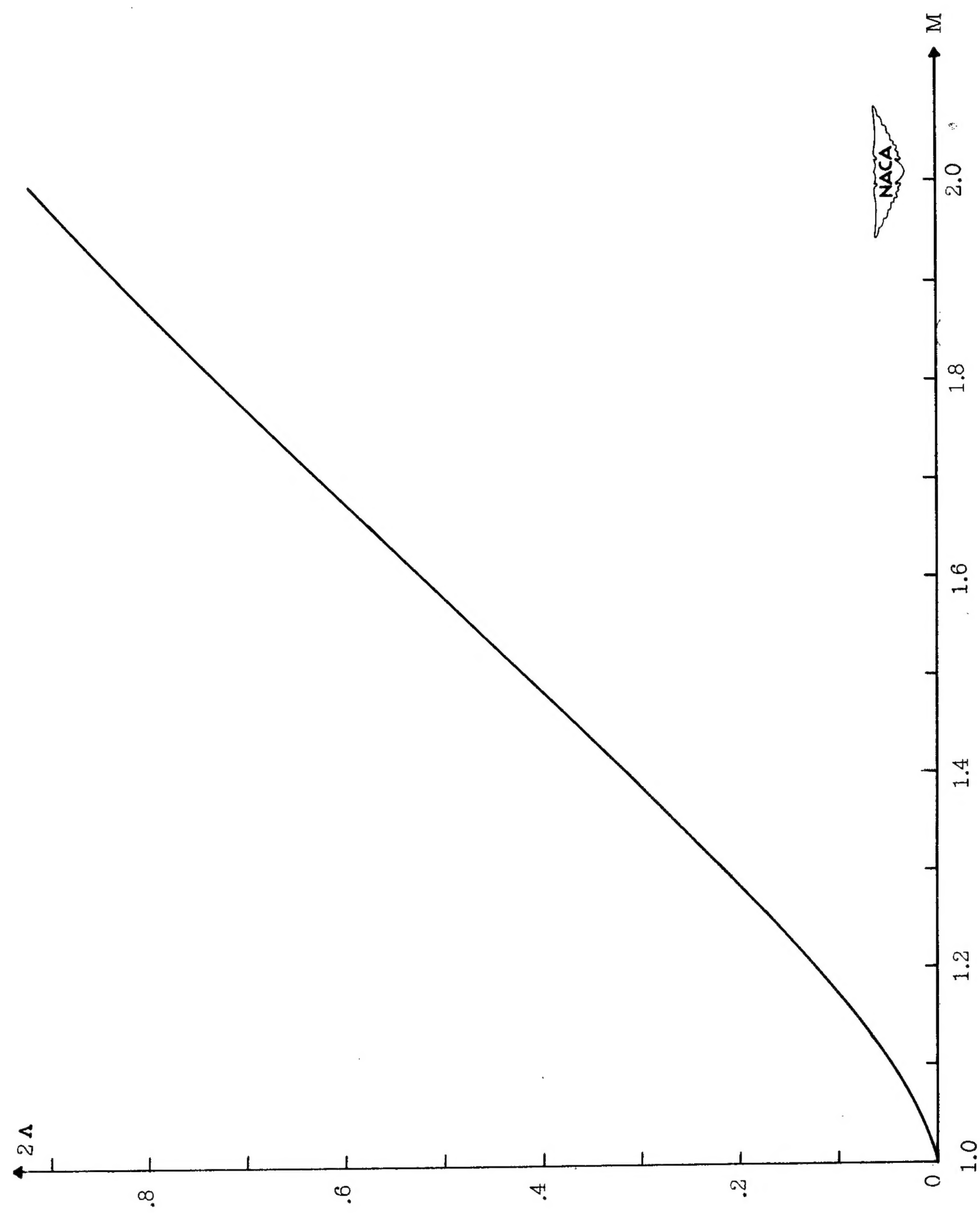

Figure 1.- Variation of 2Λ with M.

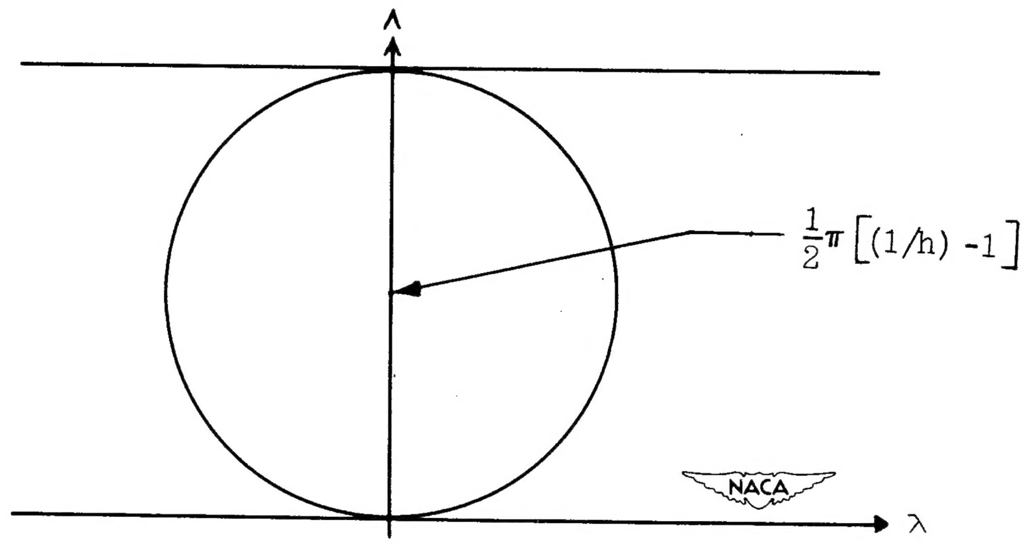

Figure 2.- The strip $-\infty < \lambda < \infty$, $0 < \Lambda < \left[(1/h) - 1\right]$ and the circle of convergence of the development of F_1 around the point $\lambda = 0$, $\Lambda = \frac{1}{2}\pi\left[(1/h) - 1\right]$.

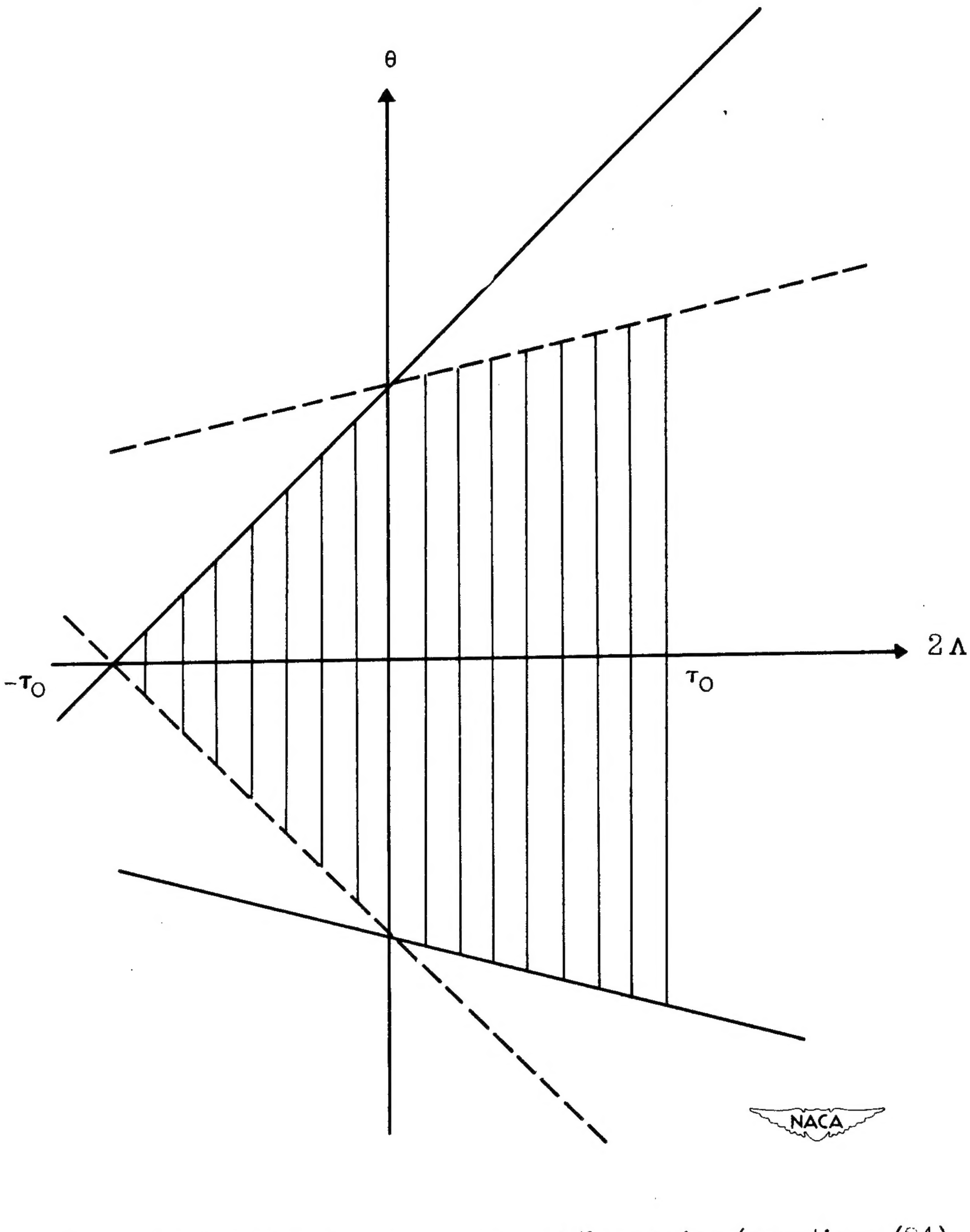

Figure 3.- The domain of convergence of the series (equations (24) and (25)).

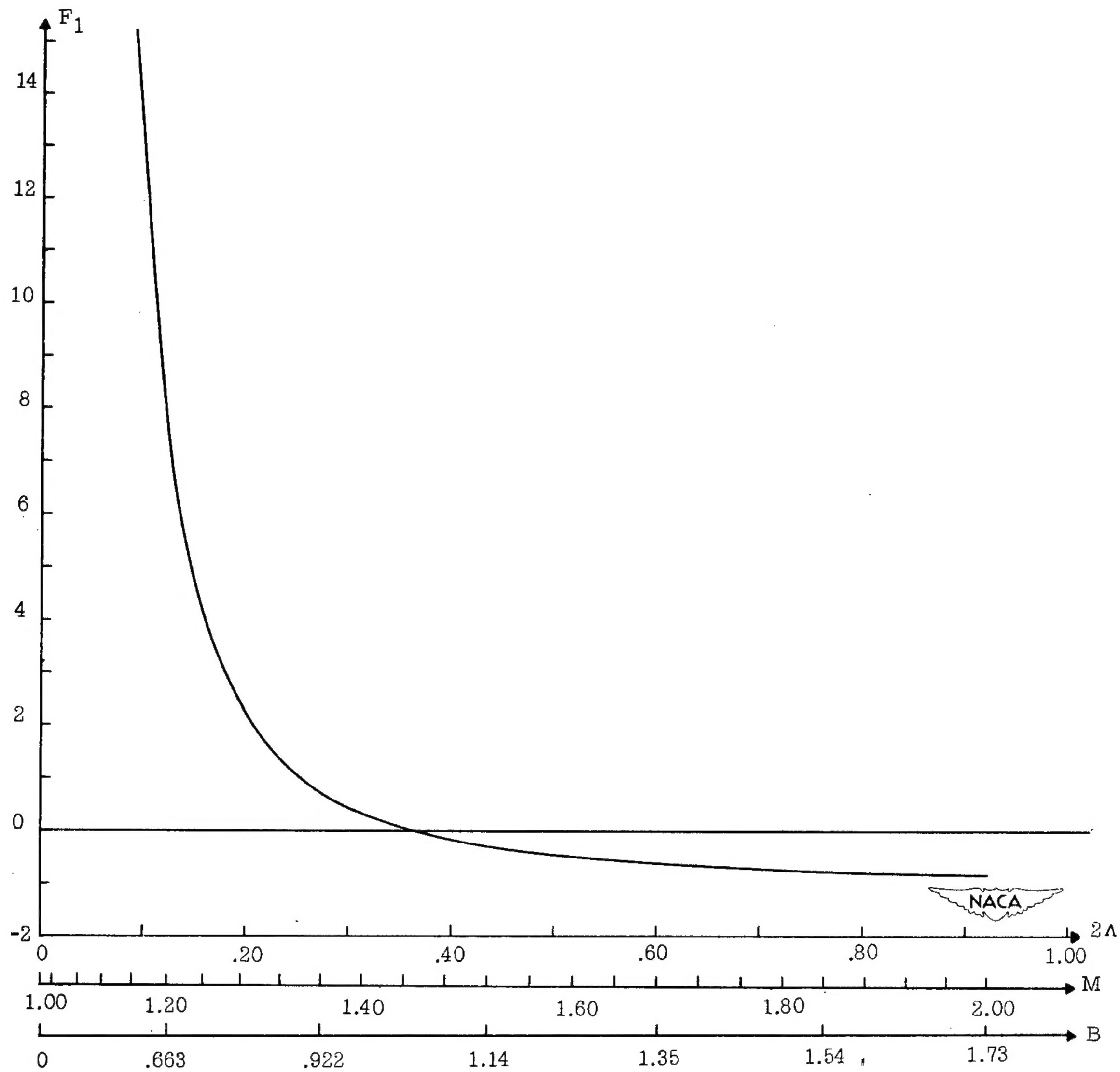

Figure 4.- F_1 as a function of M, B, and 2Λ.